

Perihelion precession and bending of light near charged dilaton black holes

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Perihelion precession of planetary orbits and bending angle of null geodesics are estimated near a charged dilaton spacetime in string inspired models. It is shown that for dilaton coupled gravity, the leading order measure in the angle of bending of light comes purely from vacuum expectation value of the dilaton field which may be envisaged as an indicator of a dominant stringy effects over the curvature effects. The dependence of the perihelion precession and bending angle with various parameters are estimated.

I. INTRODUCTION:

The first experimental verification of Einstein's general theory of relativity was done in the context of the perihelion precession of mercury orbit and bending of light by a massive star i.e. our sun during a solar eclipse ([1], [2], [3]). Such scenario assumes a Schwarzschild metric in the background spacetime. Meanwhile there have been various proposals of alternative theories of gravitation beyond Einstein's gravity. Specially in the context of string theory the effects of the extra scalar field, namely the dilaton, leads to a new model of gravity in the low energy effective theory in four dimensions. (see Gibbons et.al. ([4], [5]), Hawking ([6]), Witten ([7]), Mayers et.al. ([8], [9]) and Chandrasekhar et.al. ([10])).

In this work we first present a general formalism based on perturbative method to calculate perihelion precession and light bending for Reissner-Nordström solution, and then extend it to dilaton gravity in presence of a $U(1)$ gauge field which yields a charged black hole solution in low energy limit of string theory compactified to four dimensions. Such black hole solution has been analysed in great details by Garfinkle et.al.([11]) (see also Coleman [12]). We have shown that, while in case of pure Reissner-Nordström (R-N) solution the bending of light, in the leading order, is similar to Schwarzschild scenario, it however differs significantly for a charged dilaton black hole solution. For such a black hole solution the bending angle depends non-trivially on the vacuum expectation value of dilaton which is a measure of string coupling. Our results further reveals that while for R-N spacetime the perihelion shift decreases with increase in electric charge, for a dilaton coupled em background it increases with the electric charge.

We first present our calculation of perihelion precession and bending of light for a Reissner-Nordström black hole in the next section. We then extend our calculations for a charged dilaton black hole in the following section. The paper ends with a discussion on our results.

II. REISSNER-NORDSTRÖM BLACK HOLE:

a.Perihelion Precession:

The metric for R-N black hole is given by,(see [13], [14])

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$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})dr^2 + r^2d\Omega^2 \quad (1)$$

where M is the mass of the black hole and Q is the charge. Since the metric does not contain t and ϕ explicitly, we have a conserved energy given by,

$$p_t = -mE \quad (2)$$

Here m is the mass of the particle and E is the energy per particle mass, which are conserved. Note that in our system of units $c = 1$. Thus mass has same dimension as energy and E is dimensionless. The angular momentum is given by,

$$p_\phi = mL \quad (3)$$

Which is conserved. We therefore choose the motion of the particle in equatorial plane ($\theta = \frac{\pi}{2}$) (see [15], [16]). Thus we have

$$p_\theta = 0$$

Using the relation,

$$p_\mu p^\mu = -m^2 \quad (4)$$

We obtain

$$p^r p^r = g^{rr} p_r p^r = g^{rr} [-m^2 - p^\phi p_\phi - p_t p^t] \quad (5)$$

Defining,

$$p^r = m \frac{dr}{d\lambda} \quad (6)$$

where λ is some affine parameter connected with the proper time of the particle, one finally arrives at,

$$m^2 \left(\frac{dr}{d\lambda} \right)^2 = g^{rr} [-m^2 - g^{\phi\phi} p_\phi p_\phi - g^{tt} p_t p_t] \quad (7)$$

Using values for p_ϕ and p_t we obtain,

$$m^2 \left(\frac{dr}{d\lambda} \right)^2 = g^{rr} [-m^2 - g^{\phi\phi} m^2 L^2 - g^{tt} m^2 E^2] \quad (8)$$

Using the expression of metric co-efficient,

$$\left(\frac{dr}{d\lambda} \right)^2 = E^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \left[1 + \frac{L^2}{r^2} \right] \quad (9)$$

Now we define,

$$p^\phi = m \frac{d\phi}{d\lambda} \quad (10)$$

Thus,

$$\frac{d\phi}{d\lambda} = \frac{L}{r^2} \quad (11)$$

Using eqn.(9) and eqn.(11) we get,

$$\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 \left(\frac{d\phi}{d\lambda}\right)^{-2} \quad (12)$$

From these expressions we obtain,

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2} \left[E^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \left(1 + \frac{L^2}{r^2}\right) \right] \quad (13)$$

Now substituting $r = 1/u$ the above equation reduces to,

$$\left(\frac{du}{d\phi}\right)^2 = (u')^2 = \frac{E^2}{L^2} - (1 - 2Mu + Q^2u^2)(u^2 + \frac{1}{L^2}) \quad (14)$$

where ' denotes derivative w.r.t ϕ .

Differentiating again with respect to ϕ we obtain,

$$u'' + u = \frac{M}{L^2} - \frac{Q^2}{L^2}u + 3Mu^2 - 2Q^2u^3 \quad (15)$$

We keep terms upto order u^2 and neglect the u^3 term in the above expression. This leads to,

$$u'' + \left(1 + \frac{Q^2}{L^2}\right)u = \frac{M}{L^2} + 3Mu^2 \quad (16)$$

This equation can be rewritten as,

$$u'' + (1 + \varepsilon_1)u = A + \varepsilon u^2 \quad (17)$$

where we have taken, $A = \frac{M}{L^2}$ and $\varepsilon_1 = \frac{Q^2}{L^2}$ and $\varepsilon = 3M$.

We assume that after each revolution the period of ϕ changes by a small amount, i.e.,

$$\phi_{period} = 2\pi[1 + \alpha\varepsilon]$$

where α depends on A, ε_1 .

Using the perturbation method the solution for u is taken as,

$$u = A + B\cos(1 - \alpha\varepsilon)\phi + \varepsilon u_1(\phi) \quad (18)$$

This yields,

$$u'' = -B(1 - 2\alpha\varepsilon)\cos(1 - \alpha\varepsilon)\phi + \varepsilon u_1'' \quad (19)$$

where we have kept terms linear in ε . This is valid when GM/c^2 as well as the quantity Q^2/L^2 are small.

$$\varepsilon u^2 = \varepsilon[A^2 + B^2\cos^2(1 - \alpha\varepsilon)\phi + 2AB\cos(1 - \alpha\varepsilon)\phi] \quad (20)$$

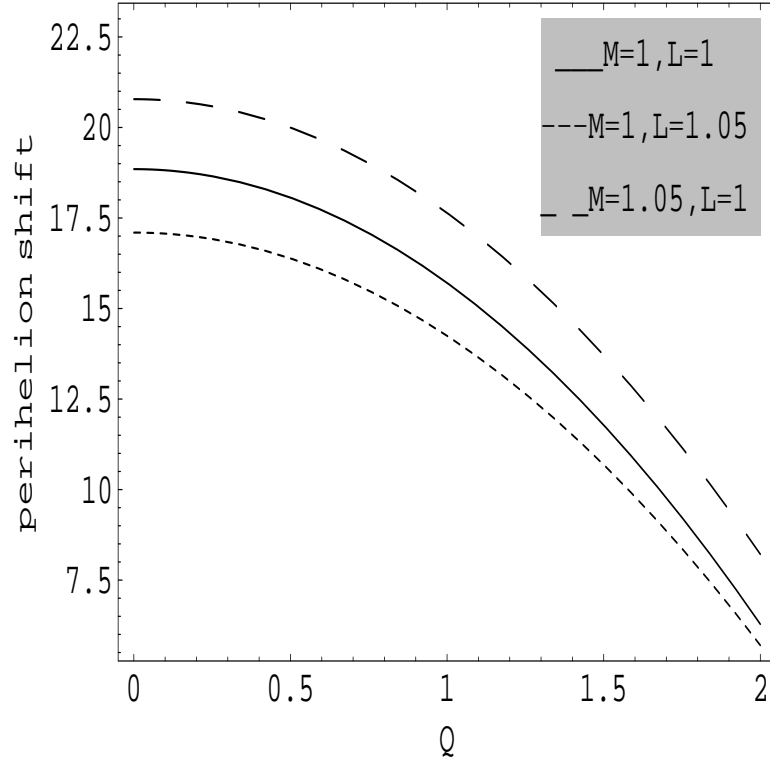


Figure 1: The figure shows variation of perihelion shift with charge for R-N black hole.

Substitution and simplification leads to,

$$u_1'' + u_1 = A^2 - A\frac{\varepsilon_1}{\varepsilon} - \frac{\varepsilon_1}{\varepsilon}B\cos(1 - \alpha\varepsilon)\phi - B^2\cos^2\phi + 2AB\cos(1 - \alpha\varepsilon)\phi - 2\alpha B\cos(1 - \alpha\varepsilon)\phi \quad (21)$$

Note that the solution of an equation of the form $u'' + u = \sum_i A_i \cos \omega_i \phi$ will be non-singular when $\cos(1 - \alpha\varepsilon)\phi$ term vanishes (see [17]), which in turn implies,

$$\alpha = A - \frac{\varepsilon_1}{2\varepsilon} \quad (22)$$

Hence the perihelion precession would be,

$$\delta\phi = 2\pi\left(A - \frac{\varepsilon_1}{2\varepsilon}\right)\varepsilon = 2\pi A\varepsilon - \pi\varepsilon_1 \quad (23)$$

Thus for R-N case shift is given by,

$$\delta\phi = -\frac{\pi Q^2}{L^2} + 6\pi \frac{M^2}{J^2} \quad (24)$$

Figure 1 depicts how the perihelion shift decreases with increase of charge of the black hole. Note that the last term is the precession for Schwarzschild case when $Q = 0$.

b. Bending of Light:

For bending of light we can write the null geodesic as,

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left[\frac{L^2}{r^2}\right] \quad (25)$$

using eqn.(11) we obtain the required expression as,

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2} \left[E^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \left(\frac{L^2}{r^2}\right) \right] \quad (26)$$

Thus finally we obtain the orbit equation which is given by,

$$\left(\frac{du}{d\phi}\right)^2 = (u')^2 = \frac{E^2}{L^2} - (1 - 2Mu + Q^2u^2)u^2 \quad (27)$$

Now defining $\frac{E^2}{L^2} = \frac{1}{j^2}$ the above expression reduces to,

$$\left(\frac{du}{d\phi}\right)^2 = (u')^2 = \frac{1}{j^2} - (1 - 2Mu + Q^2u^2)u^2 \quad (28)$$

Equation (28) indicates that the term that contains Q is of the order of u^4 and therefore has a very negligible influence on the measure of bending of light. Hence we can conclude that in R-N black hole bending of light is almost identical to that of Schwarzschild black hole if one ignores the terms beyond the leading order i.e. u^3 .

III. DILATON COUPLED ELECTROMAGNETIC FIELD

Static uncharged black hole in general relativity are described by Schwarzschild solution. If mass of the black hole is much large compared to Planck mass then this also describes to a good approximation the uncharged black hole in string theory except regions near singularity. However this is not the case for Einstein-Maxwell solutions. The dilaton coupling with F^2 implies that every solution with non zero $F_{\mu\nu}$ will come with a non zero dilaton. Thus the charged black hole solution in general relativity (which is the Reissner-nordström solution) appears in a new form in string theory due to the presence of dilaton. The effective four dimensional low energy lagrangian obtained from string theory is,

$$S = \int d^4x \sqrt{-g} [-R + e^{-2\phi} F^2 + 2(\nabla\phi)^2]$$

where $F_{\mu\nu}$ is the Maxwell field associated with a $U(1)$ subgroup of $E_8 \times E_8$ or $Spin(32)/Z_2$. We have set the remaining gauge fields and antisymmetric tensor field $H_{\mu\nu\rho}$ to zero and ϕ represents the dilaton strength (see Garfinkle et.al.[11], Coleman [12], Vega et.al. [18], Bekenstein [19] and Witten [20])

a.Perihelion Precession:

The static spherically symmetric solution would give the following line element corresponding to the above action as,(see Garfinkle et.al.[11])

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}dr^2 + r\left(r - e^{2\phi_0} \frac{Q^2}{M}\right)d\Omega^2 \quad (29)$$

where we have, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. As usual we have taken our motion in the equatorial plane such that, $d\Omega^2 = d\phi^2$. Also ϕ_0 is the asymptotic constant value of dilaton.

Note that this is almost identical to the Schwarzschild metric, only it differs by the fact that areas of spheres of constant r and t now depend on Q and are decreased from Schwarzschild values. In particular

the surface $r = \frac{Q^2 e^{2\phi_0}}{M}$ is singular. Also $r = 2M$ is the regular event horizon. We could also define a dilaton charge,

$$D = \frac{1}{4\pi} \int d^2\sigma^\mu \nabla_\mu \phi$$

where the integral is over a two sphere at spatial infinity. For charged black hole this leads to,

$$D = -\frac{Q^2 e^{2\phi_0}}{2M}$$

D is not any free parameter, once asymptotic value of dilaton strength is given it is completely determined by M and Q and is always negative.

Following the same procedure as in the previous section we find for a test particle,

$$\left(\frac{dr}{d\lambda}\right)^2 = -g^{rr}[\epsilon + g^{tt}E^2 + g^{\phi\phi}L^2] \quad (30)$$

Where E and L are conserved energy and angular momentum. Also ϵ is 0 for a massless photon and is 1 for a massive particle. Defining $A = e^{2\phi_0} \frac{Q^2}{M}$ the above equation simplifies to,

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(\epsilon + \frac{L^2}{r(r-A)}\right) \quad (31)$$

Using the fact that,

$$\frac{d\phi}{d\lambda} = g^{\phi\phi}L = \frac{L}{r(r-A)} \quad (32)$$

We arrive at,

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^2(r-A)^2}{L^2} \left[E^2 - \left(1 - \frac{2M}{r}\right)\left(\epsilon + \frac{L^2}{r(r-A)}\right)\right] \quad (33)$$

Substituting $r = \frac{1}{u}$ the above orbit equation reduces to,

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{L^2} [E^2(1-Au)^2 - \epsilon(1-2Mu)(1-Au)^2 - u^2 L^2(1-Au)(1-2Mu)] \quad (34)$$

For massive particles ($\epsilon = 1$) equation (34) becomes,

$$\left(\frac{du}{d\phi}\right)^2 = \left(\frac{E^2-1}{L^2}\right) + u \left[\frac{2(M+A)}{L^2} - \frac{2AE^2}{L^2}\right] - u^2 \left[1 + \frac{A^2+4MA}{L^2} - \frac{A^2E^2}{L^2}\right] + u^3 \left[2M+A + \frac{2MA^2}{L^2}\right] \quad (35)$$

Differentiation yields,

$$u'' + \left(1 + \frac{A^2+4MA-A^2E^2}{L^2}\right)u = \frac{M+A-AE^2}{L^2} + \frac{3}{2}u^2 \left(2M+A + \frac{2MA^2}{L^2}\right) \quad (36)$$

Note that in the limit $A = 0$ the above equation reduces to orbit equation for Schwarzschild black hole. Just as in previous case here also the quantity L/A is small since the quantity A contain mass of the black hole in inverse power.

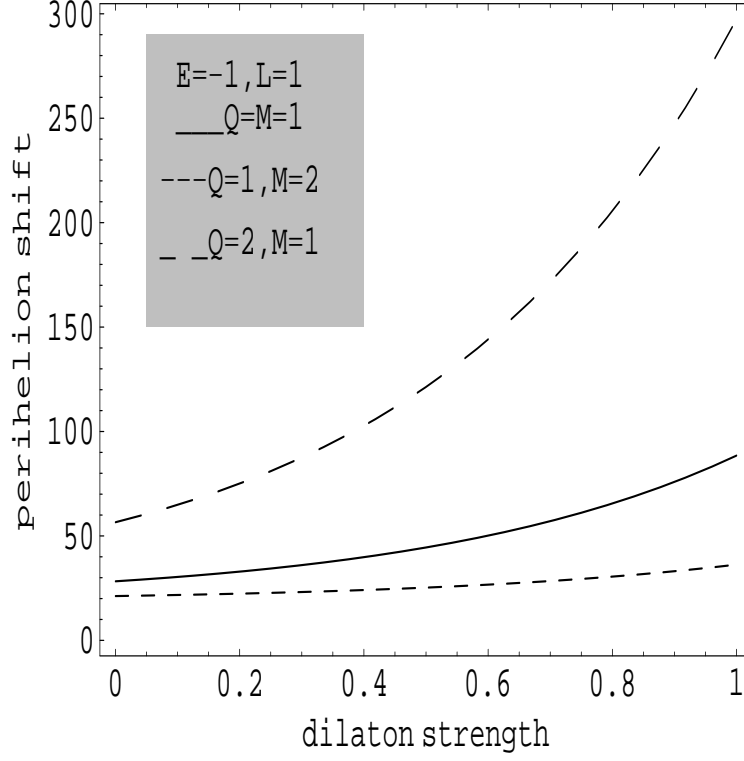


Figure 2: The figure shows variation of perihelion shift with dilaton field for different choices of Q and M for charged black hole in string theory.

Comparing with equation (17) the perihelion precession is given as,

$$\delta\phi = 6\pi \frac{M^2}{L^2} \left[1 + \frac{Q^2}{M^2} (1 - E^2) e^{2\phi_0} \right] \left[1 + \frac{Q^2}{2M^2} e^{2\phi_0} + \frac{Q^4}{M^2 L^2} e^{4\phi_0} \right] - \frac{\pi Q^2}{M L^2} e^{2\phi_0} \left[4M + \frac{Q^2}{M} (1 - E^2) e^{2\phi_0} \right] \quad (37)$$

Neglecting terms of the order of Q^4 we obtain,

$$\delta\phi = 6\pi \frac{M^2}{L^2} + 2\pi \frac{Q^2}{L^2} e^{2\phi_0} \left[\frac{5}{2} - E^2 \right] \quad (38)$$

Again for $A = 0$ it reduces to Schwarzschild case. Note that for $\phi_0 = 0$, A is not equal to zero and hence does not reduce to R-N black hole. From figure-2 we observe that perihelion shift increases exponentially with dilaton strength. It also increases with charge Q , which is quite opposite to that in R-N scenario (see figure-3). It increases with energy E (see figure-4) and reaches an asymptotic limit as $E \rightarrow 0$ when it no longer represents a bound state.

We emphasise that the perihelion shift increases very fast with increase of dilaton strength and it also increases with increase in charge of the black hole.

b. Bending of Light:

Finally the massless particle geodesic i.e. for photon the orbit eq. yields,

$$\left(\frac{du}{d\phi} \right)^2 = \frac{E^2}{L^2} (1 - 2Au + A^2 u^2) - u^2 + (A + 2M)u^3 - 2MAu^4 \quad (39)$$

Introducing $j = L/E$ i.e. angular momentum per unit energy or equivalently angular momentum per particle mass, we obtain,

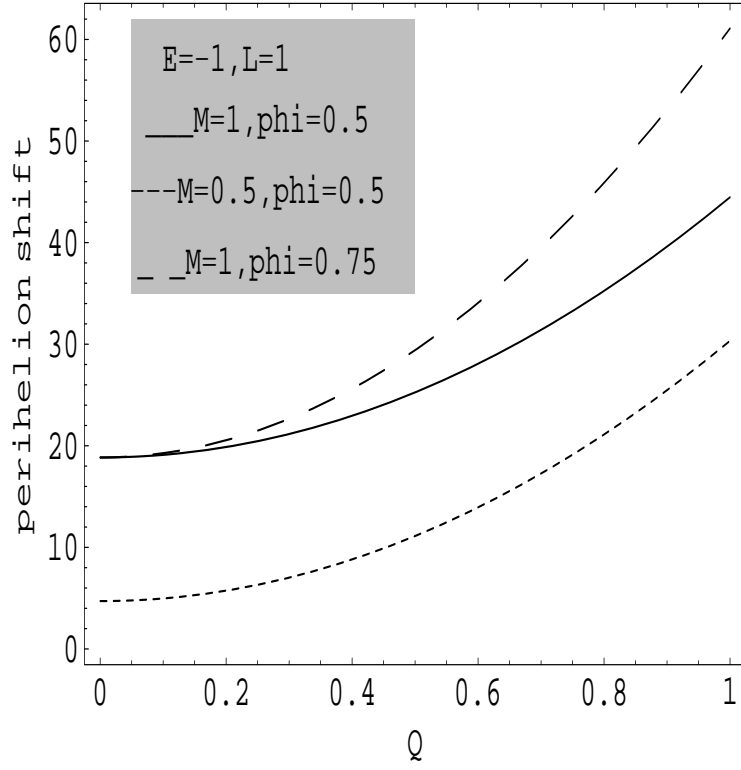


Figure 3: The figure shows variation of perihelion shift with charge for different choices of mass and dilaton parameter for charged black hole in string theory.

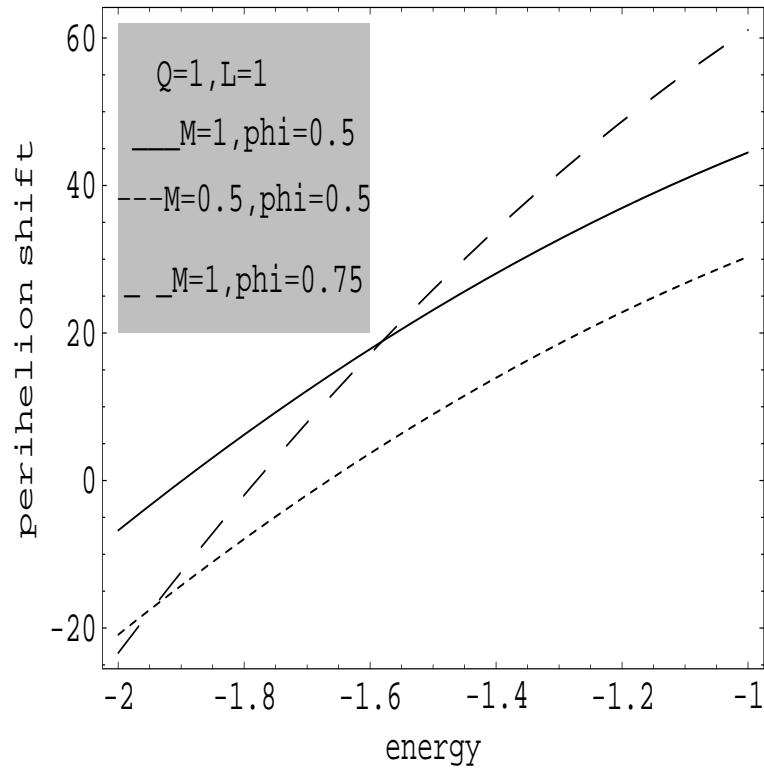


Figure 4: The figure shows variation of perihelion shift with energy for different choices of charge and dilaton parameter for charged black hole in string theory.

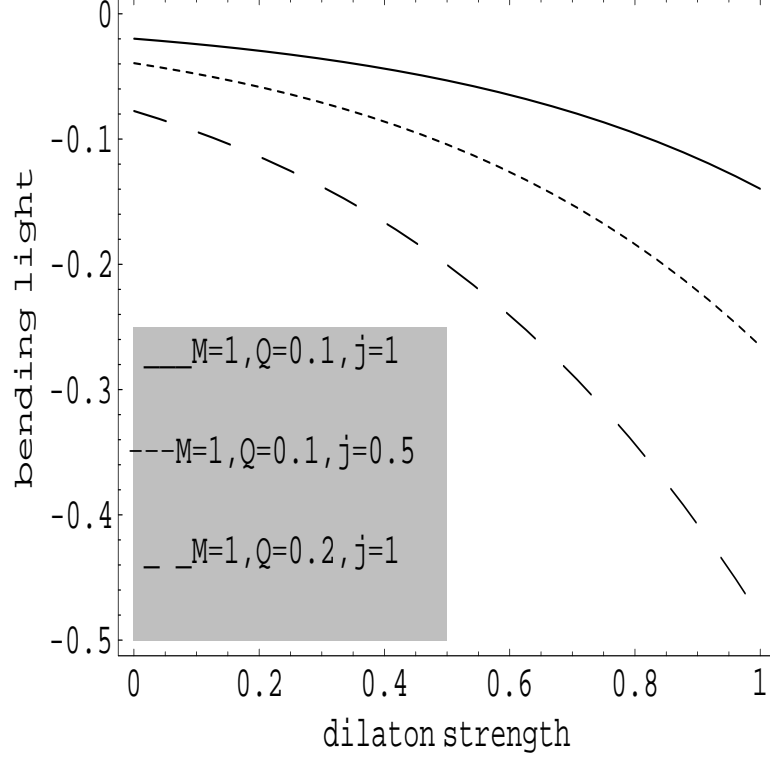


Figure 5: The figure shows variation of bending of light with dilaton strength energy for different choices of mass and charge for charged black hole in string theory.

$$\frac{d\phi}{du} = \frac{1}{\sqrt{\frac{1}{j^2} - \frac{2Au}{j^2} + u^2\left(\frac{A^2}{j^2} - 1\right) + (A + 2M)u^3}} \quad (40)$$

Neglecting terms of order u^3 the above differential equation can be integrated with impact factor nearly equal to j to yield,

$$\phi - \phi_0 = \int_{u_0}^u \frac{du}{\sqrt{\frac{1}{j^2} - c_1 u - c_2 u^2}} \quad (41)$$

where $c_1 = \frac{2A}{j^2}$ and $c_2 = (1 - \frac{A^2}{j^2})$. The lower limit u_0 represents the position of photon when it has not felt the presence of the field i.e. at $r \rightarrow \infty$. The upper limit yields the minimum distance of approach, which in this case is solution of the equation $u' = 0$. Using the above two facts we find the bending of light which is now given as,

$$\Delta\phi = \frac{2j}{\sqrt{j^2 - A^2}} \left[\frac{\pi}{2} - \sin^{-1}\left[\frac{A}{j}\right] \right] - \pi \quad (42)$$

Note that in the limit $A \rightarrow 0$ this vanishes, which is consistent since in the equation (40) we have eliminated the u^3 term which would appear in the Schwarzschild case. As we have neglected the dominant term in Schwarzschild case, we should not expect Schwarzschild result to be retrieved. Thus the above bending is dependent crucially on the dilaton asymptotic value and comes from a more dominant term than that in Schwarzschild scenario. So this bending can be interpreted as a stringy signature.

However the above solution is valid for $j^2 > A^2$. For the other choice the bending of light is given by,

$$\Delta\phi = \frac{2j}{\sqrt{j^2 - A^2}} \log \left| \frac{j^2}{A^2 - j^2 - Aj} \right| \quad (43)$$

This also vanishes in the limit $A = 0$ and hence consistent with the above discussion. The first solution is applicable when the L/E of the particle is much greater than dilaton strength, and the second condition is just the reverse of the first condition.

We reiterate that the angle for bending of light increases in magnitude with dilaton strength and also with increase with the charge.

IV. CONCLUSIONS

In this work we have considered a string inspired dilaton coupled gravitational theory and studied the role of dilaton field on the Astrophysical signatures like perihelion precession of Planetary orbits and bending of light near a dilaton coupled charged black hole. We first analyse the Reissner-Nordstrom black hole scenario and determine the charge dependence of the precession as well as bending angle. Subsequently carrying out our analysis to the dilaton coupled charged black hole we show that the dilaton charge produces a contribution to the bending angle even at a lower order than the Schwarzschild spacetime, indicating a purely stringy signature at an Astrophysical observation. Though the dilaton vacuum expectation value, which decides the magnitude of the bending angle, is suppressed by the scale of the theory however it's presence at one order lower in u than the pure Schwarzschild scenario exhibits the possibility of having a more pronounced stringy effect than the pure Einsteinian gravity. The dependence of the bending angle as well as precession angle with dilaton vacuum value has also been determined for different values of mass and charge. For precession angle the charge dependence turns out to be opposite to that in R-N scenario, indicating that the dilaton coupled charged spacetime solutions belong to a new class of solutions than the ordinary R-N scenario.

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